Theorem 3. As $\varepsilon \to +0$ the solution of the variational inequality (2.1) converges weakly to the solution of the variational inequality (2.4).

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THE SUFFICIENT CONDITIONS FOR AN EXTREMUM IN PROBLEMS OF OPTIMIZING THE SHAPES OF ELASTIC PLATES *

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The problems of selecting the thickness distributions of elastic plates in order to maximize the fundamental free vibrations frequency, as well as to minimize the strain potential energy, are considered necessary and sufficient conditions are obtained for a weak local extremum in the such optimal design problems. These conditions retain their form even for reciprocal problems: minimization of plate weight when there are constraints on the fundamental frequency or the strain potential energy. The conditions obtained include an integral estimate on the maximum growth of second derivatives of the thickness distributions that satisfy the necessary extremum conditions.

Problems on optimizing the shape of elastic plates have been solved numerically /1-8/. It has been proved /9/ that these problems cannot have a strong extemum. It is shown /10,11/ that for solutions to exists it is sufficient to improve integral constraints on the nature of the growth of the derivatives of the allowable thickness distributions.

1. Formulation of the problem. Consider a plate of variable thickness $h(x, y)$ clamped along a piecewise-smooth contour Γ bounding the domain D in the xy plane. Let S be the area of the domain D and V the volume of the plate. In the undeformed state the plate middle surface coincides with the domain D. The plate is simply supported on the part Γ_1 of the boundary Γ, and rigidly clamped on the remaining part Γ_2. The function of plate deflections is denoted by $w(x, y)$. We introduce the dimensionless variables

$$x' = xS^{-1/2}, \quad y' = yS^{-1/2}, \quad h'(x, y) = h(x, y) SV^{-1} \tag{1.1}$$

The problem of the frequencies of free vibrations has the following form in the notation used (we omit the primes on the dimensionless variables):

$$A(h) w(x, y) = \lambda h w(x, y), \quad \lambda = 12(1-\nu^2) E^{-1} S^4 V^{-2} \omega^2 \tag{1.2}$$

$$(w)_\Gamma = 0 \left(\frac{\partial w}{\partial n}\right)_{\Gamma_2} = 0, \quad \left(h^3\left(\Delta w - \frac{1-\nu}{R}\frac{\partial w}{\partial n}\right)\right)_{\Gamma_1} = 0 \tag{1.3}$$

$$A(h) = \frac{\partial^2}{\partial x^2} h^3 \left(\frac{\partial^2}{\partial x^2} + \nu \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial y^2} h^3 \left(\frac{\partial^2}{\partial y^2} + \nu \frac{\partial^2}{\partial x^2}\right) + 2(1-\nu) \frac{\partial^2}{\partial x \partial y} h^3 \frac{\partial^2}{\partial x \partial y} \tag{1.4}$$

Here E is Young's modulus, ν is Poisson's ratio, ω is the frequency of free vibrations, $\partial w/\partial n$ is the derivative with respect to the external normal to Γ, R is the radius of curvature, and Δ is the Laplace operator.

In the variables (1.1) the static bending problem of a plate loaded by a transverse force $p(x, y)$ has the form

$$A(h) w(x, y) = q(x, y), \quad q = 12(1-\nu^2) E^{-1} S^{-1/2} V^{-3} p(x, y) \tag{1.5}$$

where the differential operator $A(h)$ is given by (1.4), and the function w satisfies the

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boundary conditions (1.3).

We consider the Sobolev space $W_2^k(D)$ $(k = 0, 1, 2)$ of functions square-summable together with their derivatives to order k inclusive. Assuming that $h(x, y)$ is a function continuous in D $h(x, y) \geqslant h_1 > 0$ $(h_1 = \text{const})$, we introduce a bilinear symmetric positive-definite form /12/, generated by the operator $A(h)$

$$A_h(w,u) = \iint_D a_h(w,u)\,dx\,dy, \quad a_h(w,u) = h^3(w_{xx}(u_{xx} + \nu u_{yy}) + w_{yy}(u_{yy} + \nu u_{xx}) + 2(1-\nu) w_{xy}u_{xy}) \tag{1.6}$$

(the subscripts denote calculation of the corresponding partial derivatives with respect to x and y). The form $A_h(w, u)$ is defined and continuous in functions from the set H obtained by closure in the space $W_2^2(D)$ of the set of functions, infinitely differentiable in $\bar{D}$ and satisfying the boundary conditions (1.3).

We shall consider weak solutions $w(x, y) \in H$ of the boundary value problems (1.2), (1.5) satisfying the integral identities

$$A_h(w, u) = \lambda(hw, u), \quad A_h(w, u) = (q, u) \tag{1.7}$$

that are valid for any functions $u(x, y) \in H$. Here and henceforth, the parentheses denote the scalar product in the space $L_2(D)$. Under the assumptions made, the theorem about the discrete spectrum is valid for the eigenvalue problem, as is the theorem on the existence of a solution of the boundary value problem if $q \in H^*$, where H^* is the space conjugate to the space H /12, 13/.

We introduce additional assumptions on the nature of the possible plate thickness distributions. We let Q denote the set of functions $h(x, y)$, satisfying the conditions

$$\iint_D h(x,y)\,dx\,dy \leqslant 1, \quad 0 < h_1 \leqslant h(x,y) \leqslant h_2 \tag{1.8}$$

$$(h_1, h_2 = \text{const}, h_2 > 1) \iint_D (\partial^2 h)^2\,dx\,dy =$$

$$\iint_D \left(\left(\frac{\partial^2 h}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 h}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2 h}{\partial y^2}\right)^2\right) dx\,dy \leqslant C^2 = \text{const}$$

The last condition in (1.8) yields an integral constraint on the growth of the second derivatives of the thickness distributions. Its necessity is dictated by the following reasons. Firstly, by virtue of the Sobolev embedding theorem /13/ it ensures continuity of the function $h(x, y)$, which is a natural requirement on the nature of the thickness distribution. Secondly, it is a sufficient condition for solutions of optimization problems formulated below to exist /10-11/. Thirdly, from the viewpoint of mechanics the hypothesis of a Kirchhoff-Love rectilinear normal element should be satisfied. The absence of this latter condition in (1.8) allows the appearance of a thickness distribution with arbitrarily large values of the Gaussian curvature; it is difficult here to expect any satisfactory compliance with the hypothesis mentioned. (It is shown by a passage to the limit in /14/ that the plate equations are asymptotically exact if the period of the thickness variation T is considerably greater than the thickness h itself, i.e., $h/T \ll 1$).

Remark. Certain authors /8/ treat them as stiffness ribs in solving the optimization problem numerically without the last constraint in (1.8), and obtaining arbitrarily large thickness distribution "peaks". However, the mathematical model of a plate with stiffness ribs /15/ does not correspond to the initial equations for which these solutions have been obtained, which makes such treatment unjustified.

We will now formulate the optimal design problems.

1°. Among all thickness distributions $h(x, y) \in Q$ it is required to find the distribution for which the minimum eigenvalue of the spectral problem (1.7) will be maximal.

2°. Among all distributions $h(x, y) \in Q$ it is required to find that for which the magnitude of the strain potential energy in the boundary value problem (1.7) will be minimal.

2. Calculation of the variations of the functionals.

Let $h(x, y) \in Q$. We give the function h an increment in the form $\varepsilon\delta h(x, y)$, where ε is a fairly small number, and $\delta h(x, y)$ is a function from $W_2^2(D)$. By virtue of the conditions (1.8) the function δh is not arbitrary. However, at this stage we are interested in the dependence of the functionals of problems 1 and 2 on the increment δh without taking account of the constraints (1.8). We shall later turn to a complete formulation of the optimization problems taking all the constraints into account.

We will use the results of an analytic perturbation of the spectrum of selfadjoint operators /16, 17/ by assuming that the first eigenvalue is prime. For $h + \varepsilon\delta h$ the first eigenfunction w_1 and the first eigenvalue λ_1 of problem (1.7) can be represented in the form of an asymptotic series in powers of a small parameter

$$\lambda_1 + \varepsilon\mu_1(h,\ \delta h) + \varepsilon^2\mu_2(h,\ \delta h) + o(\varepsilon^2) \tag{2.1}$$
$$w_1(x,y) + \varepsilon v_1(x,y;\ \delta h) + \varepsilon^2 v_2(x,y;\ \delta h) + \varepsilon^3\psi_\varepsilon$$

Here ψ_ε is a function of the space $W_2^2(D)$ bounded in the norm as $\varepsilon \to 0$.

We substitute (2.1) into the first equation in (1.7) and we collect terms in identical powers of ε. We obtain integral equalities for the functions $w_1, v_1, v_2 \in H$ that are valid for any functions $u \in H$

$$B_h(w_1,\ u) = 0,\quad B_h(v_1,u) = -B_h^1(w_1,\ u) + \mu_1(hw_1,\ u),\quad B_h(v_2,\ u) = -B_h^1(v_1,\ u) - B_h^2(w_1,\ u) + \mu_1(hv_1,\ u) + \mu_1(w_1\delta h,\ u) + \mu_2(hw_1,\ u) \tag{2.2}$$
$$(B_h(w,\ u) = A_h(w,\ u) - \lambda_1(hw,\ u))$$

Here $B_h^i\ (i = 1, 2)$ are bilinear forms determined by the formula (the expression $a_h(w,\ u)$ is defined in (1.6))

$$B_h^i(w,u) = \frac{1}{i!}\iint_D \left(\frac{d^i}{d\varepsilon^i}\left(a_{h+\varepsilon\delta h}(w,u) - \lambda_1(h+\varepsilon\delta h)uw\right)\right)_{\varepsilon=0} dxdy \tag{2.3}$$

Let $\{\lambda_l\}_{l=2}^\infty$, $\{w_l(x,y)\}_{l=2}^\infty$ be the remaining eigenvalues and eigenfunctions of problem (1.7) evaluated for $h = h(x,y)$. The eigenfunctions can be normalized (δ_{ij} is the Kronecker delta) /12/:

$$(w_ih, w_j) = \delta_{ij},\quad A_h(w_i, w_j) = \sqrt{\lambda_i\lambda_j}\,\delta_{ij} \tag{2.4}$$

We set $u = w_1$ in the second equation in (2.2) and we take account of the first equation and (2.4); then

$$\mu_1(h,\ \delta h) = B_h^1(w_1,\ w_1) \tag{2.5}$$

We find an expression for the function v_1 in (2.1) by giving the first correction in ε to the eigenfunction w_1. The system of eigenfunctions $\{w_i\}_{i=1}^\infty$ is complete in H /12/, consequently, the following representation holds:

$$v_1(x,y;\delta h) = \sum_{s=1}^\infty q_s w_s(x,y) \tag{2.6}$$

We substitute the function v_1 into the second equation in (2.2) and we successively set $u = w_l\ (l = 2, 3, \ldots)$, and we obtain an expression for the coefficients of the series (2.6)

$$q_s = -(\lambda_s - \lambda_1)^{-1}B_h^1(w_1,\ w_s),\quad s = 2, 3, \ldots$$

The constant q_1 is determined from the normalization condition (2.4) and is not essential for further computations.

We set $u = w_1 \in H$ in the third equation in (2.2) and use the expression for the coefficients of the expansion (2.6) and condition (2.4). We have

$$\mu_2(h,\delta h) = B_h^2(w_1,w_1) - \sum_{s=2}^\infty \frac{(B_h^1(w_1,w_s))^2}{\lambda_s - \lambda_1} \tag{2.7}$$

We now consider the boundary value problem (1.7). For $h + \varepsilon\delta h$, its solution can be represented in the form of a series in eigenvalues of the spectral problem (1.7). Consequently, the following representation holds:

$$w(x,y) + \varepsilon z_1(x,y;\delta h) + \varepsilon^2 z_2(x,y;\delta h) + \varepsilon^3\psi_\varepsilon(x,y;\delta h)$$

where $w(x,y)$ is the solution of problem (1.7) for $h = h(x,y)$. We substitute this expansion into (1.7). We obtain integral equalities for the functions $w, z_1, z_2 \in H$ ($A_h^i = B_h^i$ for $\lambda_1 = 0$; the forms $B_h^i\ (i = 1, 2)$ are defined in (2.3))

$$A_h(w,u) = (q,u),\quad A_h(z_1,u) + A_h^1(w,u) = 0 \tag{2.8}$$
$$A_h(z_2,\ u) + A_h^1(z_1,\ u) + A_h^2(w,\ u) = 0 \qquad \forall u \in H$$

The functional of the problem, the strain potential energy, is given by the formula

$$U(h) = (q,\ w) = (A(h)w,\ w) \tag{2.9}$$

consequently, the first correction in ε to the value of the functional (2.9) equals (q, z_1).

Setting $u = w$ in the second equation of (2.8), we have $A_h(z_1,\ w) = A_h(w,\ z_1) = (q, z_1) = -A_h^1(w,\ w)$ from the first equality in (2.8). Hence

$$\delta U(h) = -A_h^1(w,\ w) \tag{2.10}$$

To calculate the second correction in ε we set $u = w \in H$ in the third equation in (2.8), then $A_h(z_2, w) = -A_h^1(z_1, w) - A_h^2(w, w)$. Now setting $u = z_1 \in H$ in the second equation in (2.8), we have $A_h^1(w, z_1) = A_h^1(z_1, w) = -A_h(z_1, z_1)$. In sum, we obtain an expression for the second variation of the functional (2.9)

$$\delta^2 U(h) = (z_2, g) = A_h(z_2, w) = A_h(z_1, z_1) - A_h^2(w, w) \tag{2.11}$$

Remark. The formulas obtained for the variations of the functionals are weak functional derivatives according to Gateaux. Later the property of strong Fréchet differentiability of the functionals is utilized. As a rule, these derivatives are in agreement for traditional calculus of variations problems. For the case of the strain potential energy functional, the agreement between these derivatives follows from the results in /10, 11/. For the functional of the prime eigenvalue the proof of agreement between the weak and strong derivatives is based on the property of the continuous dependence of the eigenfunctions and eigenvalues on the elements $h \in Q$.

3. Necessary conditions for an extremum. We introduce the functions $\sigma^2(x, y) = h_2 - h(x, y)$, $\tau^2(x, y) = h(x, y) - h_1$ by considering σ and τ as new controls in problems 1° and 2° of Sec. 1.

We consider first the case of the spectral problem (1.7). We form the expanded Lagrange functional

$$L(h, \sigma, \tau) = -\lambda_1(h) + \varkappa_1(h, 1) + \varkappa_2(\partial^2 h, \partial^2 h) + \tag{3.1}$$
$$\iint_D \theta_1(x, y)(h(x, y) - h_1 - \tau^2(x, y))\,dx\,dy + \iint_D \theta_2(x, y)(h_2 - h(x, y) - \sigma^2(x, y))\,dx\,dy$$

$\varkappa_1, \varkappa_2 = \text{const} \geqslant 0$, θ_1, θ_2 are elements from the space $W_2^{-2}(D)$ conjugate to the space $W_2^2(D)$ /13/ (the expression $(\partial^2 h)^2$ is defined in (1.8)). The necessary conditions for the extremum are satisfaction of the conditions /18/

$$\delta_h L = 0, \quad \delta_\sigma L = 0, \quad \delta_\tau L = 0 \tag{3.2}$$

$$\varkappa_1((h, 1) - 1) = 0, \quad \varkappa_2((\partial^2 h, \partial^2 h) - C^2) = 0 \tag{3.3}$$

Here $\delta_h L, \delta_\sigma L, \delta_\tau L$ are the first variations of the functional (3.11) with respect to the controls h, σ, τ.

Taking account of (2.5), we obtain from (3.2)

$$\delta_h L = -\mu_1(h, \delta h) + \varkappa_1(\delta h, 1) + 2\varkappa_2(\partial^2 h, \partial^2 \delta h) + \tag{3.4}$$
$$(\theta_1 - \theta_2, \delta h) = 0$$
$$(\partial^2 h, \partial^2 \delta h) = \iint_D (h_{xx}\delta h_{xx} + 2h_{xy}\delta h_{xy} + h_{yy}\delta h_{yy})\,dx\,dy$$
$$\theta_1(x, y)\,\tau(x, y) = 0, \quad \theta_2(x, y)\,\sigma(x, y) = 0$$

Here and henceforth, the subscripts denote evaluation of the corresponding partial derivatives.

Suppose $\sigma(x, y) \neq 0$ and $\tau(x, y) \neq 0$, then $\theta_1(x, y) = \theta_2(x, y) = 0$, i.e. $h_1 < h(x, y) < h_2$. We will denote the set of such points $(x, y) \in D$ by D_0.

Let $\sigma(x, y) \neq 0$ and $\tau(x, y) = 0$; then $\theta_2(x, y) = 0$ and $h(x, y) = h_1$.

If $\sigma(x, y) = 0$, and $\tau(x, y) \neq 0$, then $\theta_1(x, y) = 0$ and $h(x, y) = h_2$.

We denote the set of such points $(x, y) \in D$ by $D_{\min}$ and $D_{\max}$, respectively.

The case when $\sigma(x, y) = \tau(x, y) = 0$ is impossible since $h_1 < h_2$. We apply Green's formula (/19/, p. 109) (Γ_0 is the boundary of the domain D_0)

$$(\partial^2 h, \partial^2 \delta h) = (\delta h, \Delta\Delta h) + \int_{\Gamma_0} \Delta h \frac{\partial \delta h}{\partial n}\,ds - \int_{\Gamma_0} \left(\frac{\partial}{\partial n}\Delta h\right)\delta h\,ds$$

($\Delta\Delta$ is the biharmonic operator). Let the conditions

$$(\Delta h)_{\Gamma_0} = 0 \quad \left(\frac{\partial}{\partial n}\Delta h\right)_{\Gamma_0} = 0 \tag{3.5}$$

be satisfied, which corresponds to smooth emergence of the thickness distribution $h(x, y)$ and the upper and lower constraints h_2 and h_1. Taking account of (3.3), (3.5) and formula (2.5), we write the necessary condition in the form ($b_h^1 = a_h^1 - \lambda_1 w_1^2$, $a_h^1 = da_h/dh$, the form a_h is defined in (1.6))

$$-b_h^1(w_1, w_1) - \varkappa_1 + 2\varkappa_2\Delta\Delta h = \theta_2 - \theta_1 \tag{3.6}$$

Together with conditions (3.5), Eq.(3.6) is a boundary value problem in the function $h(x, y)$ whose solution should be understood in the weak sense, i.e., as the integral identity (3.4) which holds for any functions $\delta h \in W_2^2(D)$.

The non-positivity of the elements $\theta_1(x, y)$ and $\theta_2(x, y)$ from $W_2^{-2}(D)$ results from a second-order necessary condition requiring the non-negativity of the second variations of the functional (3.1) with respect to σ and τ /18/. Then conditions (3.6) can be written in the form

$$b_h^1 + \varkappa_1 \geqslant 0, \quad (x, y) \in D_{\min}, \quad h = h_1 \tag{3.7}$$
$$b_h^1 + \varkappa_1 \leqslant 0, \quad (x, y) \in D_{\max}, \quad h = h_2$$
$$b_h^1 + \varkappa_1 - 2\varkappa_2 \Delta\Delta h = 0, \quad (x, y) \in D, \quad h_1 < h(x, y) < h_2$$
$$b_h^1(w_1, w_1) = a_h^1(w_1, w_1) - \lambda_1 w_1^2$$

where the function h satisfies conditions (3.5) on the boundary of the domain D_0.

Analogous conditions can also be written in the case of the problem for minimum potential energy. They have the form (3.7) when $b_h^1(w_1, w_1)$ is replaced by $a_h^1(w_1, w_1)$.

Remark. Conditions (3.7) are obtained under assumptions on the regularity of the extremal problem formulated /18/. If the regularity condition is not satisfied, then either $\varkappa_1 = \varkappa_2 = 0$ or $\varkappa_1 = 0$ and $h(x, y) = \text{const}$.

4. Sufficient conditions for an extremum.

We formulate the main result.

Theorem 1. For $h = h(x, y) \in Q$ let the necessary conditions for an extremum (3.7), (3.3) be satisfied in the problem of maximizing the first eigenfrequency, where the first two inequalities in (3.7) are satisfied as strict inequalities. Then the function $h(x, y)$ achieves a weak local maximum of the problem if the constant C in the integral constraint (1.8) satisfies the estimate $C^2 < h_1^2(h_2 - h_1)2\gamma$ in the class of variations satisfying the condition $(\partial^2\delta h, \partial^2\delta h) < \infty$, where γ is a constant dependent only on the geometry of the domain D.

Proof. The sufficient conditions for a weak local maximum /18/ are the positive-definiteness of the second variations of the functional (3.1) in the variations δh for which the following is satisfied (the expressions $(\partial^2 h, \partial^2\delta h)$ are defined in (3.4))

$$(\delta h, 1) = 0, \quad (\partial^2 h, \partial^2\delta h) = 0, \quad (\theta_1 - \theta_2, \delta h) = 0 \tag{4.1}$$

From the positive-definiteness of the second variations in σ and τ we have $\theta_1(x, y) > 0$, $\theta_2(x, y) > 0$. Consequently, the inequalities (3.7) are satisfied as strict inequalities in the optimal solution $h(x, y)$ and it follows from (3.4) that $h(x, y) = h_1$ on $D_{\min}$ and $h(x, y) = h_2$ in $D_{\max}$. Therefore, it is sufficient to confirm the positive-definiteness of the second variation in h just in the variations $\delta h_0 \in W_2^2(D)$ which vanish in the set $D_{\min} \cup D_{\max}$. It hence follows /13/ that $\delta h_0 = (\delta h_0)_x = (\delta h_0)_y = 0$ on the boundary of the domains D_0 and $D_{\min}, D_{\max}$. We use (2.7) by noting that the second term in (2.7), taken with a minus sign, will be non-negative since $\lambda_s > \lambda_1$, $s = 2, 3, \ldots$. We have

$$\delta_h^2 L \geqslant -B_h^2(w_1, w_1) + \varkappa_2 \iint_D (\partial^2\delta h_0)^2\, dx\, dy \tag{4.2}$$

The following estimate holds (later the maximum is taken in the domain D_0, $a_h^2 = d^2 a_h/dh^2$)

$$B_h^2(w_1, w_1) = (a_h^2(w_1, w_1), \delta h_0^2) \leqslant \max|\delta h_0|^2 \iint_D a_h^2(w_1, w_1)\, dx\, dy \leqslant \tag{4.3}$$
$$6h_1^{-2}\max|\delta h_0|^2 \iint_D a_h(w_1, w_1)\, dx\, dy = 3\lambda_1 h_1^{-2}\max|\delta h_0|^2$$

Here the inequality $h/h_1 \geqslant 1$ is used, as are also the relationship $a_h(w_1, w_1)h^{-2} = a_h^2(w_1, w_1)/6$ that results from the definition of the forms a_h and a_h^2 in (1.6), and the equality $A_h(w_1, w_1) = \lambda_1$ follows from (1.7).

From the Sobolev embedding theorem /13/, the estimate ($\|\delta h_0\|_2$ is the norm of the element δh in the space $W_2^2(D)$)

$$\max|\delta h_0|^2 \leqslant \gamma_1 \|\delta h_0\|_2^2$$

follows with a fixed constant γ_1 dependent only on the geometry of the domain D_0.

Using the Friedrichs and Poincaré inequalities /12/ and conditions (4.1), it can be proved that the following estimate holds

$$\max|\delta h_0|^2 \leqslant \gamma \iint_D (\partial^2\delta h_0)^2\, dx\, dy, \quad \gamma = \text{const} > 0$$

By using the last inequality we have from (4.2) and (4.3)

$$\delta_h^2 L \geqslant (\varkappa_2 - 3\gamma h_1^{-2}\lambda_1) \iint_D (\partial^2\delta h_0)^2\, dx\, dy$$

For this expression to be positive-definite we must have

$$\varkappa_2 > 3\gamma\lambda_1 h_1^{-2} \tag{4.4}$$

From (4.4) we obtain the upper bound for values of the constant C in the constraints (1.8). For this we integrate the last equation in (3.7) over the domain D_0, taking (3.5) into account. We have (S_0 is the measure of the set D_0)

$$S_0\varkappa_1 = \iint\limits_{D_0} b_h{}^1 (w_1, w_1)\, dx\, dy$$

We multiply (3.7) by h and integrate it over the domain D_0 taking (3.5) into account. We obtain

$$2\varkappa_2 \iint\limits_{D_0} (\partial^2 h)^2\, dx\, dy = \iint\limits_{D_0} hb_h{}^1 (w_1, w_1)\, dx\, dy - \varkappa_1 \iint\limits_{D_0} h\, dx\, dy \tag{4.5}$$

If $\varkappa_2 > 0$, it then follows from (3.3) that

$$\iint\limits_{D_0} (\partial^2 h)^2\, dx\, dy = C^2 \tag{4.6}$$

On the other hand, the inequality $h/h_2 \leqslant 1$ holds, and consequently

$$\varkappa_1 S_0 \geqslant h_2^{-1} \iint\limits_{D_0} hb_h{}^1 (w_1, w_1)\, dx\, dy$$

We have from (4.5), (4.6), and the estimate (4.4)

$$C^2 \leqslant \frac{h_2^2}{6\gamma\lambda_1} \iint\limits_{D} hb_h{}^1 (w_1, w_1)\, dx\, dy \left(1 - S_0^{-1} h_2^{-1} \iint\limits_{D_0} h\, dx\, dy\right)$$

We obtain from the definition of the form $b_h{}^1 (w_1, w_1)$ in (3.7)

$$b_h{}^1 (w_1, w_1) = a_h{}^1 (w_1, w_1) - \lambda_1 w_1^2 \leqslant a_h{}^1 (w_1, w_1)$$

We use the equalities (1.6) and (2.3) defining the forms a_h and $a_h{}^1$. Then $ha_h{}^1 (w_1, w_1) = 3a_h (w_1, w_1)$ and

$$\iint\limits_{D_0} hb_h{}^1 (w_1, w_1)\, dx\, dy \leqslant 3 \iint\limits_{D} a_h (w_1, w_1)\, dx\, dy = 3\lambda_1$$

Finally, we use the estimate

$$S_0^{-1} \iint\limits_{D_0} h\, dx \geqslant h_2$$

Finally, we have the estimate mentioned in Theorem 1.

Theorem 2. For $h(x, y) \in Q$ in the problem of minimizing the functional (2.9), let the necessary conditions for an extremum (3.7), (3.3) be satisfied with $-b_h{}^1 (w_1, w_1) = a_h{}^1 (w_1, w_1)$. The inequalities in (3.7) are here satisfied as strict inequalities. Then the function $h(x, y)$ achieves a weak local minimum for the problem formulated if the constant C in the constraints (1.8) satisfies the estimate presented in the conditions of Theorem 1.

The proof of Theorem 2 is analogous to the proof of Theorem 1.

The results obtained ensure the existence of a weak extremum in the problems considered if the constant C in (1.8) is sufficiently small, i.e., if the curvature of the surface $h = h(x, y)$ changes sufficiently smoothly. They enable us to explain the discrepancy in the optimization process for large ratios h_2/h_1 /8/. In this case, the condition imposed in Theorems 1 and 2 on the maximal growth of the derivatives of the thickness distribution that satisfy the necessary condition for an extremum may be violated, and it is impossible to guarantee an optimum at stationary points.

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STOCHASTIC BIFURCATION IN THE THEORY OF THE FLEXURE OF SPHERICAL SHELLS AND CIRCULAR MEMBRANES*

S.I. VOLKOV

The capacity of rigidly clamped elastic membranes and open shallow spherical shells of circular outline that are in equilibrium under the action of a radial stress, given uniformly on the contour, and transverse loads distributed radially along the surface to form a field with a quasi-Gaussian probability measure to retain shape is investigated. It is assumed that the behaviour of the membranes and shells is described by von Karman equations taken in a radial approximation.

The following method /1/ is used. A generalization of the probability density, a probability functional (PF) induced by the probability measure of the load and the operator of the problem is constructed in the space of possible solutions of the initial boundary value problem (the concept of probability density in the functional space of individual realizations of a random field of the desired parameters was first utilized in statistical hydromechanics problems /2/). The times of a substantial change in the shape or an abrupt decrease in the shell (and membrance) carrying capacity are related to the first bifurcation of the PF modes with respect to the growth of the compressive force.

The application of this method starts with the derivation of the equations for the PF extremals in the space of weighted derivatives of the deflection function with respect to the dimensionless variable radius. Within the framework of the Galerkin method, solutions of the designated equation are determined. Simple relationships are determined that relate the radial stresses to the statistical characteristics of the transverse load field at the time of bifurcation of these solutions. It is shown that up to the time of the first bifurcation of PF has just one extremal, a trivial mode for the membranes but a non-trivial mode for the shells. Then by starting with the time mentioned the membrane PF reaches maxima on the extremals bifurcating from the trivial, while the shell PF acquires a new maximum (in addition to the existing maximum) on still another

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